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Energy Profile Fluctuations in Dissipative Nonequilibrium Stationary States

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The exact large deviation function (ldf) for the fluctuations of the energy density field is computed for a chain of Ising (or more generally Potts) spins driven by a zero-temperature (dissipative) Glauber dynamics and sustained in a nontrivial stationary regime by an arbitrary energy injection mechanism at the boundary of the system. It is found that this ldf is independent of the dynamical details of the energy injection, and that the energy fluctuations, unlike conservative systems in a nonequilibrium state, are not spatially correlated in the stationary regime.

KEY WORDS: Large deviations; nonequilibrium stationary states; dissipative systems; Glauber dynamics; Potts models.

1. INTRODUCTION

In a series of recent papers, Derrida *et al.*[1] studied the nonequilibrium stationary states of exactly solvable models characterized by *conservative* inner dynamics: particles diffuse in a one dimensional chain of sites without annihilation, and are described at the hydrodynamical length- and time-scales by a Fick's (diffusion) law (with or without a systematic drift force). The authors succeeded in exactly computing the nonequilibrium free energy (associated with a given density profile) which displayed very interesting features: for instance, the nonequilibrium situation induces long range correlations which make the free energy nonadditive. The authors discovered however an elegant and quite unusual sub-additivity principle whose generality for other models is still an open question. Their results also showed that although the second moment of density fluctuations is the same as

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previously found with fluctuating hydrodynamics arguments,[2] a quadratic (gaussian) approximation is not sufficient for higher cumulants.

In this paper, we address the same question of characterizing of nonequilibrium stationary states (NESS), but for dissipative inner dynamics. Such situations are widely encountered: for instance, the granular matter continuously shaked by a piston can be considered for long time scales as being in a NESS; the velocity field of a turbulent flow is also subjected to a dissipative (Navier–Stokes) equation and can be sustained in a stationary state as well. These dissipative systems are altogether different from conservative ones, since no concept of thermodynamical equilibrium can be applied to them: in absence of energy input, the full rest is the ultimate state of the system; as a result, no pertubative strategy can be deployed to describe their physics, and in fact the nonequilibrium stationary states of dissipative systems are not well known (see for instance[3,4] and references therein).

In particular, at the light of the recent works aforementioned, natural questions arise for dissipative NESS: are they similarly characterized by a high level of correlation? are the fluctuations to some extent insensitive to the details of the inner dynamics and/or the injection mechanism? is the notion of nonequilibrium "thermodynamical potential" even relevant?

To answer these questions, and to develop further the ideas presented in refs. 5,6 we consider here a one-dimensional dissipative model for which some exact results on the energy structuration can be extracted: our model is a semi-infinite 1D chain of Ising spins (easily generalized to a Potts model) subjected to zero-temperature Glauber dynamics. These intrinsically dissipative dynamics are supplemented with an arbitrary (Poissonian to be simple) flipping process of the first spin as a way to inject energy into the system.

Our results can be summarized as follows: (i) the concept of nonequilibrium potential can be extended via a nonhomogeneous space coarsening of this dissipative system; (ii) this potential is *independent* of the nature of the injection mechanism, and reflects mainly the dynamical inner self organization of the system; (iii) this potential *does not* display correlations, i.e. it is "additive", as soon as a stretching of the space is taken into account; (iv) such a potential cannot be defined for any dissipative system, and the conditions that system dynamics must fulfill to have a global observable obeying the large deviation theorem are not clear.

The paper is organized as follows. In a first part we define the model and its dynamics; then we give some preliminary results concerning the stationary state and introduce the cumulative energy E. Afterwards we compute the large deviation function of (ldf) E and the ldf associated with an energy profile. We end up with a discussion of the results.

2. DEFINITION OF THE MODEL

The model we consider is a one-dimensional semi-infinite chain of Ising spins $\sigma_0, \sigma_1, \ldots, \sigma_j, \ldots$ (a finite chain could be considered as well). The spins follow zero-temperature Glauber dynamics, except for the spin σ_0, which flips according to a Poisson process of parameter λ : the probability associated with a flip of spin j from a configuration $\mathcal{C}$ between t and $t+dt$ is $w(\mathcal{C} \to \mathcal{C}_j)dt = [1 - \sigma_j(\sigma_{j+1} + \sigma_{j-1})/2] \times dt$ if $j \neq 0$ (the starting configuration is termed $\mathcal{C}$ and the j-flipped configuration $\mathcal{C}_j$) and $\lambda\, dt$ for σ_0. As a result, the master equation describing dynamics in the model is

$$\partial_t P(\mathcal{C}) = -\sum_{j \geqslant 0} w(\mathcal{C} \to \mathcal{C}_j) P(\mathcal{C}) + \sum_{j \geqslant 0} w(\mathcal{C}_j \to \mathcal{C}) P(\mathcal{C}_j) \tag{1}$$

$$= -\lambda[P(\mathcal{C}) - P(\mathcal{C}_0)] - \sum_{j \geqslant 1} [1 - \sigma_j(\sigma_{j+1} + \sigma_{j-1})/2] P(\mathcal{C}) + \sum_{j \geqslant 1} [1 + \sigma_j(\sigma_{j+1} + \sigma_{j-1})/2] P(\mathcal{C}_j) \tag{2}$$

In the last equality, the spin values refer obviously to configuration $\mathcal{C}$.

Inner Glauber dynamics are essentially dissipative: the domain walls (which are elementary energy excitations) move randomly and annihilate by pairs when colliding. Thus, in absence of energy input, any initial condition would eventually relax to a state characterized by the same value of all spins; in the model considered here, such an external input is provided by the Poissonian motion of the spin labeled 0, which gives energy to the system when σ_0 flips from σ_1 to $-\sigma_1$. As a result, after a transient, a stationary state takes place which is described in the following.[2]

3. THE STATIONARY STATE

3.1. Mean Injected Power

If the summation $\sum_{\mathcal{C}} \sigma_0 \sigma_i(\ldots)$ is made on the dynamical Eq. (1), one gets

$$\partial_t \langle \sigma_0 \sigma_i \rangle = -2(\lambda + 1)\langle \sigma_0 \sigma_i \rangle + \langle \sigma_0 \sigma_{i+1} \rangle + \langle \sigma_0 \sigma_{i-1} \rangle \tag{3}$$

When the stationary state is assumed, and according to $\sigma_0^2 = 1$, (3) leads to $\langle \sigma_0 \sigma_i \rangle = [1 + \lambda - \sqrt{\lambda^2 + 2\lambda}]^i$ (this correlation vanishes when $i \to \infty$).

[2] we are aware that the stationary state of the infinite system needs in general a diverging time to be established, but this problem can be circumvented by a clever choice of the initial conditions.

From this calculation an interesting physical result can be deduced: the (mean) power P_{inj} injected by the spin 0 inside the system is related to $\langle\sigma_0\sigma_1\rangle$, since between t and $t+dt$, the energy ceded is in average $2\lambda\,dt\times$ $[\text{Prob}(\sigma_0=\sigma_1)-\text{Prob}(\sigma_0=-\sigma_1)]$, whence[7]

$$P_{\text{inj}}=2\lambda\langle\sigma_0\sigma_1\rangle \tag{4}$$

$$=2\lambda\left(1+\lambda-\sqrt{\lambda^2+2\lambda}\right) \tag{5}$$

The injected power is an increasing function of λ which saturates to 2 for large λ; this value is physically dictated by internal dynamics, i.e. the ability of the system to diffuse into the bulk the energy excitations created at the boundary. It is to note that there is thus no notion of "optimal" time scale concerning the energy injection, which is not evident *a priori*: one could imagine that an optimal waiting time could leave domain walls move away from the boundary.

3.2. Two-points Correlations and Energy Density Profile

Another interesting quantity describing the stationary state is the average energy density profile

$$\langle e_n\rangle=1-\langle\sigma_n\sigma_{n+1}\rangle \tag{6}$$

This quantity is more complicated to obtain. To this end, it is useful to interpret Glauber dynamics as coalescing paths dynamics:[8] the update of any spin is equivalent to a random choice (Poissonian with parameter 1) of the spin value among one of its two neighbours, such that the value $\sigma_n(t)$ can be traced back in time from spin to spin until either the time origin or the boundary σ_0 is reached at a certain spin index or time (respectively). If the stationary state is studied, the notion of time origin is irrelevant, and any "path of constant spin value" eventually reaches the zeroth spin (see Fig. 1). If two spins σ_n and σ_p are now monitored, two paths emerge from n and p, are traced back in time, and possibly coalesce if they meet each other before reaching the boundary. They are statistically independent, since the flipping processes in two distinct sites are not correlated. As a result, the average $\langle\sigma_n\sigma_p\rangle$ can be calculated as follows: consider two random walkers starting at sites n and p. If they meet before touching the site 0, this occurence of the paths gives a factor 1 in the computation of $\langle\sigma_n\sigma_p\rangle$. Otherwise, the n-path (resp. p) arrives at site zero at time t_n (resp. t_p), and this occurence gives a factor $\langle\sigma_0(t_n)\sigma_0(t_p)\rangle=$ $\exp(-2\lambda|t_n-t_p|)$, where the average is on the Poissonian process σ_0. As a

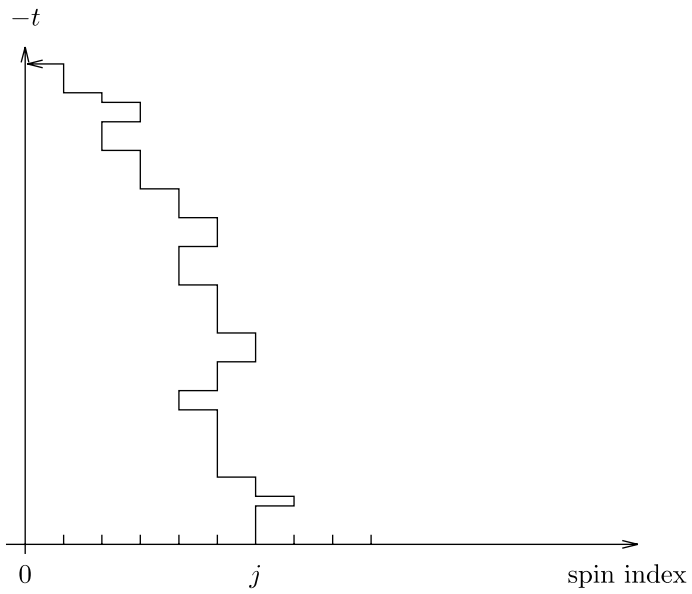

Fig. 1. The value of $\sigma_j(t)$ can be traced back in time from neighbouring spin to neighbouring spin until it reaches σ_0 whose statistical properties are known.

result original dynamics are mapped on dynamics of two random walkers plus dynamics of σ_0. These considerations can be summarized as

$$\langle \sigma_n \sigma_p \rangle = c_{np} + \int_0^\infty dt_n \int_0^\infty dt_p \text{Prob}[t_n, t_p \text{ and } (n, p)] \exp(-2\lambda |t_n - t_p|) \quad (7)$$

In this expression, c_{np} is the probability that the walkers starting at n and p meet each other before reaching the zeroth site (we will also use $c_{n,p} = 1 - c_{np}$; generally, a comma is put between two indices when the associated walkers are supposed to avoid each other; conversely the absence of comma holds for coalescing walkers); besides Prob$[t_n, t_p$ and $(n, p)]$ is the probability that the n-walker and the p-walker reach site 0 at times t_n and t_p respectively without having met each other beforehand.

These expressions can be exactly computed. Indeed, consider two independent random walkers starting at n and p. There are *a priori* three possible situations (Fig. 2): either (i) they do not cross at all, or (ii) they cross and "exchange" their arrival time, or (iii) they cross without exchanging their arrival time. It is easy to see that each situation of type (ii) can be mapped to a situation of the type (iii), and vice-versa (Fig. 3).

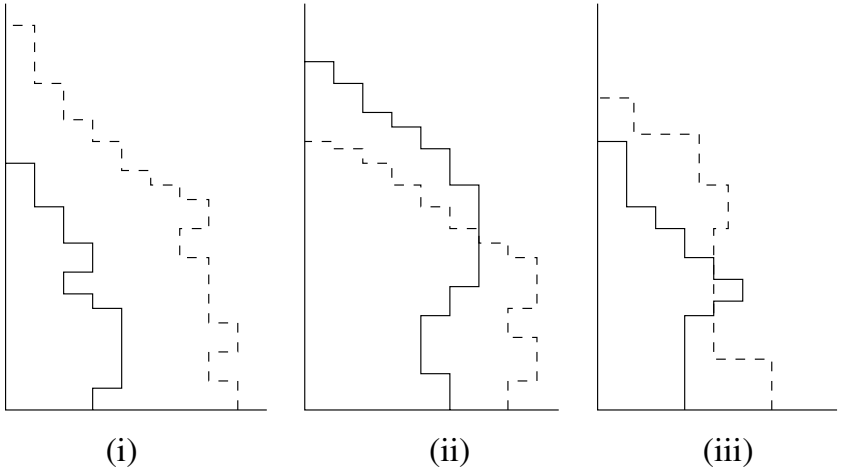

Fig. 2. Three possible generic situations for two independent walkers.

Moreover, the probability is of course conserved by the mapping, since the two paths are of the same statistical nature. As a result, if $n < p$:

$$c_{n,p} = \sum_{\text{type (i) and (iii) paths}} \text{Prob(path)} - \sum_{\text{type (ii) paths}} \text{Prob(path)} \tag{8}$$

$$= \int_0^\infty dt_n \int_{t_n}^\infty dt_p [P_n(t_n) P_p(t_p) - P_n(t_p) P_p(t_n)] \tag{9}$$

$$c_{np} = 2 \int_{t_n < t_p} P_n(t_p) P_p(t_n) \tag{10}$$

where $P_n(\tau)$ is the probability (density) for a walker initially ($t = 0$) at n to reach the site 0 *for the first time* at $t = \tau$. We get, for $n \leqslant p$:

$$\langle \sigma_n \sigma_p \rangle = 2 \int_0^\infty dt \int_t^\infty dt' P_n(t') P_p(t) + \int_0^\infty dt \int_t^\infty dt' [P_n(t) P_p(t') - P_n(t') P_p(t)] \exp(-2\lambda |t - t'|) \tag{11}$$

We are going to see that the dominant contribution to $\langle \sigma_n \sigma_p \rangle$ in the limit $n \to \infty$ is given by the first term only. This fact has important consequences on the structure of the stationary regime.

The probability $P_n(t)$ is obtained as a solution of a Brownian motion with an absorbing boundary at site 0: the probability $p(m, t|n)$ that a Brownian walker starting at site n at $t = 0$ be at site m at time t without

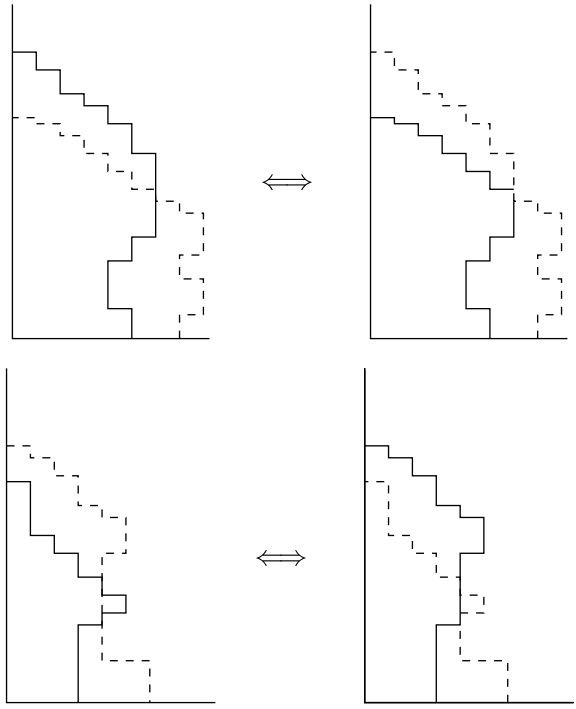

Fig. 3. Mapping of type (ii) to type (iii) paths.

having touched the zeroth site obeys the dynamical equation $\partial_t p(m|n) = p(m+1|n) + p(m-1|n) - 2p(m|n)$ if $m > 1$ and $\partial_t p(1|n) = p(2|n) - 2p(1|n)$. The probability density $P_n(t)$ is simply given by $P_n(t) = p(1, t|n)$ (actually $P_n(t)$ is $p(1, t|n)$ times the transition rate associated to the jump $1 \to 0$, but this rate were chosen equal to one). That $p(m, t|n) = p_f(m-n, t) - p_f(m+n, t)$ (with p_f is the free-boundary Brownian motion starting at site 0) is the solution is directly verified, whence one gets

$$P_n(t) = 2\int_0^1 dq\, e^{-4t\sin^2\pi q} \sin(2\pi q)\sin(2\pi qn) \tag{12}$$

With this expression, one can recast c_{np} in

$$c_{np} = 4\int_0^{1/2} dq\, \cot(\pi q)\sin(2\pi qn)\left(2\sin^2\pi q + 1 - 2\sqrt{\sin^4\pi q + \sin^2\pi q}\right)^p \tag{13}$$

An asymptotic expansion leads to the following result, valid for $n \to \infty$, $p > n$:

$$c_{np} \simeq \frac{4}{\pi} \mathrm{Atan}\left(\frac{n}{p}\right) \tag{14}$$

The second term of (11) can be estimated using the asymptotic behaviour of P_n:

$$P_n(n\tau) \underset{n\to\infty}{\simeq} \frac{1}{\sqrt{2\pi n}} \frac{1}{\tau(4\tau^2+1)^{1/4}} \times \exp n\left(\sqrt{4\tau^2+1} - 2\tau - \log\left[\frac{1}{2\tau} + \sqrt{\frac{1}{4\tau^2}+1}\right]\right) \tag{15}$$

$$\underset{\tau\gg 1}{\simeq} \frac{1}{\sqrt{4\pi n\tau^3}} \exp\left(-\frac{n}{4\tau}\right) \tag{16}$$

A cumbersome computation yields for $n \to \infty$ and $p > n$:

$$\int_0^\infty dt \int_t^\infty dt' \, [P_n(t)P_p(t') - P_n(t')P_p(t)] \exp(-2\lambda|t-t'|) \simeq \frac{24}{\pi\lambda^2} \frac{np(p^2-n^2)}{(n^2+p^2)^4} \tag{17}$$

It is important to remark that in the asymptotic limit $n \to \infty$, the dominant contribution to $\langle \sigma_n \sigma_p \rangle$ is given by the "coalescing term" c_{np}, which is *independent* of the value of λ; moreover the dominant contribution is (besides the constant 1) *nonsummable* and leads to a logarithmic divergence, whereas the λ-dependent corrections are of order $1/n^5$ (for $n - p = O(1)$) and therefore obviously summable.

These results can be applied to the energy density profile:

$$\langle e_n \rangle \simeq \frac{2}{\pi n} + O\left(\frac{1}{n^5}\right) \tag{18}$$

Once again the leading term of the energy density profile is independent of λ and nonsummable.

3.3. Energy Density Correlations

Once we have the averaged profile $\langle e_n \rangle$, it is natural to subsequently ask for the correlation function of e_n:

$$\langle e_n e_p \rangle - \langle e_n \rangle \langle e_p \rangle = \langle \sigma_n \sigma_{n+1} \sigma_p \sigma_{p+1} \rangle - \langle \sigma_n \sigma_{n+1} \rangle \langle \sigma_p \sigma_{p+1} \rangle \tag{19}$$

The 4-point correlator $\langle \sigma_n \sigma_{n+1} \sigma_p \sigma_{p+1} \rangle$ can be written $\langle \sigma_n \sigma_{n+1} \sigma_p \sigma_{p+1} \rangle = c_{nn+1pp+1} + c_{nn+1,pp+1} + \chi(n, p)$ where $c_{nn+1pp+1}$ is the probability that four walkers starting at n, $n+1$, p and $p+1$ coalesce before reaching the boundary; $c_{nn+1,pp+1}$ (note the comma) the probability that the walkers "n" and "$n+1$" on one side, walkers "p" and "$p+1$" on the other side coalesce and that the two groups reach the boundary separatly; and $\chi(n, p)$ is the term arising from situations where one or several walkers reach the boundary without coalescence. This $\chi(n, p)$ is obviously λ-dependent and is composed of several terms, each corresponding to a particular scenario for the walkers: with natural notations, $\{n, n+1, p, p+1\}$, $\{(n\ n+1\ p), p+1\}$, $\{n, (n+1\ p\ p+1)\}$, $\{(n\ n+1), p, p+1\}$, $\{n, (n+1\ p), p+1\}$, $\{n, n+1, (p\ p+1)\}$ are the different possible situations where λ plays a role. Let us consider the first term:

$$\{n, n+1, p, p+1\} = \int_{t_1<t_2<t_3<t_4} \text{Prob}[t_1, t_2, t_3, t_4 \text{ and } (n, n+1, p, p+1)] \times \langle \sigma_0(t_1) \sigma_0(t_2) \sigma_0(t_3) \sigma_0(t_4) \rangle \tag{20}$$

(the probability term refers to four walkers starting from the four sites considered and reaching without coalescence the site 0 at times $t_1 \cdots t_4$ respectively). This integral must be approximately evaluated; its dominant term comes from regions in the (t_1, t_2, t_3, t_4) plane where $t_1 \simeq t_2$ and $t_3 \simeq t_4$: in these regions only is the 4-point correlator in σ_0 significantly different from zero. As a result,

$$\{n, n+1, p, p+1\} \sim \lambda^{-2} \int_{t_1<t_2} \text{Prob}[t_1, t_1+\lambda^{-1}, t_2, t_2+\lambda^{-1} \text{ and } (n, n+1, p, p+1)] \tag{21}$$

$$\lesssim \lambda^{-1} \int_{t_1} \text{Prob}[t_1, t_1+\lambda^{-1} \text{ and } (n, n+1)] \times \lambda^{-1} \int_{t_2} \text{Prob}[t_2, t_2+\lambda^{-1} \text{ and } (p, p+1)] \tag{22}$$

But as

$$\lambda^{-1}\int_0^\infty dt\ \text{Prob}[t, t+\lambda^{-1}\ \text{and}\ (n, n+1)]$$

$$\sim \lambda^{-1}\int dt\, P_n(t)\, P_{n+1}(t+\lambda^{-1}) - P_n(t+\lambda^{-1})\, P_{n+1}(t) \tag{23}$$

$$\sim \lambda^{-2}\int_0^\infty dt\, P_n(t)^2 \left(\frac{P_{n+1}(t)}{P_n(t)}\right)' \sim \frac{\lambda^{-2}}{n^5} \tag{24}$$

we conclude that $\{n, n+1, p, p+1\}$ gives a very rapidly decreasing contribution, and we can notice besides that $\sum_{n<p}\{n, n+1, p, p+1\} < \infty$. The situation is similar if other λ-dependent terms are considered. For instance,

$$\{(n\ \ n+1\ \ p), p+1\} \sim \lambda^{-1}\int_0^\infty dt \text{Prob}[t, t+\lambda^{-1}\ \text{and}\ ((n\ \ n+1\ \ p), p+1)] \tag{25}$$

$$\lesssim \lambda^{-1}\int_0^\infty dt \text{Prob}[t, t+\lambda^{-1}\ \text{and}\ (n, p+1)]$$

$$\sim \lambda^{-2} pn \frac{p^2-n^2}{(p^2+n^2)^4} \tag{26}$$

(note the similarity between this approximation and the exact limit of (17)). Once again this term is very small and verifies $\sum_{n<p}\{(n\ \ n+1\ \ p), p+1\} < \infty$.

A special attention must be paid to the term $\{n, n+1, (p\ \ p+1)\}$. Actually it is not summable: $\sum_{n<p<N}\{n, n+1, (p\ \ p+1)\}$ diverges with a contribution $O(N)$ and another $O(\log N)$. But this term is partially "disconnected" and these divergences are exactly balanced in (19) by a term coming from $\langle\sigma_n\sigma_{n+1}\rangle\langle\sigma_p\sigma_{p+1}\rangle = (c_{nn+1} + \{n, n+1\})(c_{pp+1} + \{p, p+1\})$: actually $\{n, n+1\}c_{pp+1}$ is the counterterm which kills these "hybrid" divergences.

So, the correlation function of the energy can be recast as

$$\langle e_n e_p\rangle - \langle e_n\rangle\langle e_p\rangle = c_{nn+1\,pp+1} + c_{nn+1,\,pp+1} - 1 + c_{n,n+1} + c_{p,p+1} - c_{n,n+1}c_{p,p+1} + \tilde{\chi}(n, p) \tag{27}$$

where $\tilde{\chi}(n, p)$ depends on λ and verifies $\sum_{n<p}\tilde{\chi}(n, p) < \infty$. We will see in the following that

$$c_{nn+1\,pp+1} + c_{nn+1,\,pp+1} = 1 - c_{n,p+1} + c_{n,p} + c_{n+1,p+1} - c_{n,n+1} - c_{n+1,p} - c_{p,p+1} + c_{n,n+1}c_{p,p+1} + c_{n,p+1}c_{n+1,p} - c_{n,p}c_{n+1,p+1} \tag{28}$$

whence finally

$$\langle e_n e_p\rangle - \langle e_n\rangle\langle e_p\rangle = -c_{n,p+1} + c_{n,p} + c_{n+1,p+1} - c_{n+1,p} + c_{n,p+1}c_{n+1,p} - c_{n,p}c_{n+1,p+1} + \tilde{\chi}(n,p) \tag{29}$$

Local correlations are then easily deduced from (14):

$$\langle e_n e_{n+k}\rangle - \langle e_n\rangle\langle e_{n+k}\rangle \underset{n \text{ large},\ k\to\infty}{\simeq} -\frac{32n(n+1)(2n+1)}{3\pi^2}\frac{1}{k^5} \tag{30}$$

$$\langle e_n e_{nk}\rangle - \langle e_n\rangle\langle e_{nk}\rangle \underset{n \text{ large},\ k\to\infty}{\simeq} -\frac{32(n+1)(2n+1)}{3\pi^2}\frac{1}{n^4k^5} \tag{31}$$

and $\langle e_n^2\rangle - \langle e_n\rangle^2 \simeq 4/(\pi n)$. Thus, the energy distribution has anticorrelations which decrease with the distance. The second scaling is clearly much more relevant to describe the system, a fact that will become clearer in the following.

3.4. The Cumulative Energy *E*

It has been seen that the local structure of the stationary energy field is of course affected by the injection mechanism. Nevertheless the preceding preliminary computations highlighted a very important "boundary layer" phenomenon. To precise this, let us define the *cumulative energy* E:

$$E = \sum_{n=0}^{N-1}(1 - \sigma_n\sigma_{n+1}) \tag{32}$$

which is nothing but the energy of the N first spins. The average of E and its fluctuations share the same following properties:

- they are $O(\log N)$ when $N \to \infty$
- their leading term is independent of the injection mechanism.

To see this, let us remind first that $\langle E\rangle \sim \frac{2}{\pi}\log N$ (from Eq. (18)). As to the fluctuations, the preceding section allows to write

$$\Delta E^2 \equiv \langle E^2 \rangle - \langle E \rangle^2 \tag{33}$$

$$\underset{N\to\infty}{\sim} 2 \sum_{0\leqslant n<p<N} \Big[-c_{n,p+1} + c_{n,p} + c_{n+1,p+1} - c_{n+1,p} + c_{n,p+1}c_{n+1,p} - c_{n,p}c_{n+1,p+1} \Big] + \sum_{0\leqslant n<N} \big[\langle e_n^2 \rangle - \langle e_n \rangle^2 \big] \tag{34}$$

$$\underset{N\to\infty}{\sim} \frac{8}{\pi} \left(1 - \frac{2}{\pi} \right) \log N \tag{35}$$

The key point of this paper is to remark that these properties are actually shared by all cumulants of the distribution of E. We do not intend to give a demonstration for this, but owing to the arguments presented in the preceding subsection, this conjecture is reasonable: in an event of $2m$ walkers starting from sites located asymptotically far from the boundary, the probability associated with scenarios where one or several pairs of walkers reach the boundary is either negligible or cancelled out by the cumulant structure of the mean.

From the physical point of view, this property reflects a boundary layer structure: the injection mechanism provides energy into the system which spreads out quite efficiently despite the dissipation mechanism; however the dissipation is strongly efficient near the boundary and smoothes out the "memory" of the injection details. In other words the dissipation sets up an autosimilarity regime for the energy, which is independent of the amount transfered to the system.

From the analytical point of view, this property has two important consequences. First, we shall see that there is a ldf f associated with E, namely

$$\exists\ f,\ \lim_{N\to\infty} \frac{\log P(E/\log N = \zeta)}{\log N} = f(\zeta) \tag{36}$$

and secondly this ldf is *independent* of the injection mechanism, for the cumulants of E are directly related to the Taylor expansion coefficients of f. As already noticed,[5,9] here again the ldf captures the essential features of an observable and fades away some "irrelevant" details: the summation process tends to "universalize" the different possible behaviours in some way. Moreover, this property allows the analytical computation of $f(\zeta)$, which is the subject of the subsequent section.

4. THE LARGE DEVIATION FUNCTION OF *E*

As f does not depend on the injection mechanism, we will consider henceforth the limit $\lambda \to \infty$ (extremely quick flipping of σ_0). To get f, we will compute first the characteristic function $G(\mu) = \langle \exp(-\mu E) \rangle$, which is the Laplace transform of the probability $P(E)$; we will show that $G(\mu)$ is dominated by a term $\exp[g(\mu) \log N]$. Once g is known, the Laplace inversion formula shows that $f(\zeta)$ is given by a Legendre transformation:

$$f(\zeta) = \max_{\mu} (g(\mu) + \mu\zeta) \tag{37}$$

provided that no analyticity breaking of the prefactors of the exponentials in G occurs at the saddle point in the μ space (see ref. 5 for instance). We will assume in the following that no such problem arises, which is the most often case.

Using a classical trick of Ising spin systems and defining $\tau = (\exp(-2\mu) - 1)/2$, one can write G as

$$G(\mu) = \left\langle \prod_{i=0}^{N-1} e^{-\mu(1-\sigma_i\sigma_{i+1})} \right\rangle = \left\langle \prod_{i=0}^{N-1} [1 + \tau(1 - \sigma_i\sigma_{i+1})] \right\rangle \tag{38}$$

$$= 1 + \tau \sum_i \langle 1 - \sigma_i\sigma_{i+1} \rangle + \tau^2 \sum_{i<j} \langle (1 - \sigma_i\sigma_{i+1})(1 - \sigma_j\sigma_{j+1}) \rangle + \tau^3 \ldots \tag{39}$$

$$\equiv 1 + \tau S_1 + \tau^2 S_2 + \tau^3 S_3 + \ldots \tag{40}$$

In the limit $\lambda \to \infty$, all the averages in the preceding sum can be interpreted as probabilities related to walkers. For instance, $\langle 1 - \sigma_i\sigma_{i+1} \rangle$ is just the probability $c_{i,i+1}$ that two walkers emerging from i and $i+1$ do not meet before reaching the site 0; similarly $\langle (1 - \sigma_i\sigma_{i+1})(1 - \sigma_j\sigma_{j+1}) \rangle$ is (for $i \neq j$) the probability that during the wandering of four walkers emerging from sites labeled i, $i+1$, j, $j+1$, walkers i and $i+1$ from one side *and* walkers j and $j+1$ from the other side have not collapsed (this interpretation is also valid if $i + 1 = j$). We will denote this probability $p_{I,J}$ in the following. We will see that the probabilities $p_{I,J,K,\ldots}$ and therefore the terms S_i can be expressed in terms of the $c_{i,j}$ only.

4.1. The First Terms

The term S_1 is simply $S_1 = \sum_i c_{i,i+1}$, where the summation ranges from $i = 0$ to $i = N - 1$. The term S_2 can also be expressed in terms of the

$c_{i,j}$. To this end, let us consider $p_{I,J}$. The event (I, J), which is "walkers i and $i+1$ do not collapse, as well as walkers j and $j+1$" is equivalent to the event "$(i, i+1, j, j+1)$ or $(i, (i+1\,j), j+1)$", such that (remind that a comma holds for "have not met" whereas an absence of comma for "have met")

$$p_{I,J} = c_{i,i+1,j,j+1} + c_{i,i+1\,j,j+1} \tag{41}$$

This probability can be transformed slightly. We have the important relations (actually generalizable to any situation)

$$\begin{aligned} c_{i,jk,\ell} + c_{i,j,k\ell} + c_{i,j,k,\ell} &= c_{i,j,\ell} \\ c_{i,j,k\ell} + c_{i,j,k,\ell} &= c_{i,j,k} \end{aligned} \tag{42}$$

which are valid whatever $i \leqslant j \leqslant k \leqslant \ell$ (essentially they express the fact that if one enumerates correctly what can happen to an emerging path – all remaining paths being besides fixed – and sum the corresponding probabilities, one gets the probability of the event whence that path is eliminated). They show how to express $c_{\ldots}$ where some indices "stick" together, in terms of $c_{\ldots}$ with no sticking.

This yields for $p_{I,J}$ and S_2:

$$p_{I,J} = c_{i,i+1,j,j+1} + c_{i,i+1,j+1} - c_{i,i+1,j} \tag{43}$$

$$S_2 = \sum_{i<j} c_{i,i+1,j,j+1} + \sum_{i} c_{i,i+1,N} \tag{44}$$

The complexity of the terms increases very fast with the number of indices:

$$p_{I,J,K} = c_{i,i+1,j,j+1,k,k+1} + c_{i,i+1\,j,j+1,k,k+1} + c_{i,i+1,j,j+1k,k+1} + c_{i,i+1\,j,j+1k,k+1} \tag{45}$$

$$S_3 = \sum_{i<j} c_{i,i+1,j,j+1} + \sum_{i<j<k} c_{i,i+1,j,j+1,k,k+1} + \sum_{i<j} c_{i,i+1,j,j+1,N} \tag{46}$$

(the sum S_3 is obtained after a quite lengthy computation!). The next order requires some patience, and one gets

$$\begin{aligned} S_4 = &\sum_{i<j<k<\ell} c_{i,i+1,j,j+1,k,k+1,\ell,\ell+1} + 2\sum_{i<j<k} c_{i,i+1,j,j+1,k,k+1} \\ &+ \sum_{i<j<k} c_{i,i+1,j,j+1,k,k+1,N} + \sum_{i<j<k} c_{i,i+1,j,j+1,N} \end{aligned} \tag{47}$$

No evident regularity emerges from these first terms. It is worth noting that a rewriting of these terms is possible, since it is possible to write a term $c_{\dots}$ with odd number of indices as a sum of $c_{\dots}$ with even number of indices, but this rewriting makes the situation all but more transparent.

4.2. Resummation of *G*

A resummation of $G(\mu)$ is nevertheless possible. To do this, it is convenient to introduce a schematic representation of the terms appearing in the S_j. With evident definitions, we introduce the notations $C^{(j)}$ and $C_N^{(j)}$:

$$S_1 = C^{(1)} \tag{48}$$

$$S_2 = C^{(2)} + C_N^{(1)} \tag{49}$$

$$S_3 = C^{(3)} + C^{(2)} + C_N^{(2)} \tag{50}$$

$$S_4 = C^{(4)} + 2C^{(3)} + C_N^{(3)} + C_N^{(2)} \tag{51}$$

We will show that a recursion relation exists between S_{n+2}, S_{n+1} and S_n. The starting point is, besides the remark that S_n is a sum of terms $C^{(j)}$ and $C_N^{(j)}$, the following relation, obtained along similar lines as Eq. (42):

$$p_{I,J,K} = p_{I,J \text{ and } j+1,k,k+1} + p_{I,J \text{ and } j+1k,k+1} \tag{52}$$

$$= p_{I,J \text{ and } j+1,k+1} - p_{I,J \text{ and } j+1,k} + p_{I,J \text{ and } j+1,k,k+1} \tag{53}$$

In this relation, $p_{I,J \text{ and } j+1,k}$ terms the probability that i and $i+1$, j and $j+1$, $j+1$ and k do not meet pair by pair. The other terms have similar definitions. Moreover, we particularize the recursion relation on $p_{I,J,K}$, but it is evident that a such relation is also true if some capital indices are implied at the left of I: this relation is true at any order actually. As a result,

$$\sum_{i<j<k} p_{I,J,K} = \sum_{i<j} p_{I,J \text{ and } j+1,N} + \sum_{i<j<k} p_{I,J \text{ and } j+1,k,k+1} \tag{54}$$

Naively, we are tempted to deduce from this relation the formal law to pass from S_n to S_{n+1}: "put an extra N on one side, increment the rank on an other, and add". This rule is in fact not simple to implement for a recursion, since one does not know *a priori* how to interpret terms like $(C_N^{(n)})_N$ and $(C_N^{(n)})_{n\to n+1}$. Let us see how to do this. Again, we examplify the demonstration on a certain rank, but it is obviously valid for any.

We have

$$S_2 = \sum_{i<j} p_{I,J} \tag{55}$$

$$S_3 = \sum_{i<j<k} [p_{I,J \text{ and } j+1,k+1} - p_{I,J \text{ and } j+1,k}] + p_{I,J \text{ and } j+1,k,k+1} \tag{56}$$

So, the passage from S_n to S_{n+1} is obtained by the formal linear operator: $\mathcal{L} = (\dots)_{k+1} - (\dots)_k + (\dots)_{k,k+1}$. Let us write now S_2 in terms of $C^{(2)}$ and $C_N^{(1)}$:

$$S_2 = \sum_{i<j} \left[p_{I \text{ and } i+1,j+1} - p_{I \text{ and } i+1,j}\right] + \sum_{i<j} p_{I \text{ and } i+1,j,j+1} \tag{57}$$

The question now is: can we apply the operator $\mathcal{L}$ term by term to the Eq. (57)? We have

$$\begin{aligned} \mathcal{L}[p_{I \text{ and } i+1,j+1}] &= p_{I \text{ and } i+1,j+1,k+1} - p_{I \text{ and } i+1,j+1,k} \\ &\quad + p_{I \text{ and } i+1,j+1,k,k+1} \end{aligned} \tag{58}$$

$$\begin{aligned} \mathcal{L}[p_{I \text{ and } i+1,j}] &= p_{I \text{ and } i+1,j,k+1} - p_{I \text{ and } i+1,j,k} \\ &\quad + p_{I \text{ and } i+1,j,k,k+1} \end{aligned} \tag{59}$$

$$\begin{aligned} \mathcal{L}[p_{I \text{ and } i+1,j,j+1}] &= p_{I \text{ and } i+1,j,j+1,k+1} - p_{I \text{ and } i+1,j,j+1,k} \\ &\quad + p_{I \text{ and } i+1,j,j+1,k,k+1} \end{aligned} \tag{60}$$

On the other hand, the generic term of S_3 (cf. Eq. (56)) can be expanded "inside" using the relations:

$$p_{I,J \text{ and } j+1,k+1} = p_{I \text{ and } i+1,j,j+1,k+1} + p_{I \text{ and } i+1j,j+1,k+1} \tag{61}$$

$$= p_{I \text{ and } i+1,j,j+1} - p_{I \text{ and } i+1,j,k+1} + p_{I \text{ and } i+1,j+1,k+1} \tag{62}$$

$$p_{I,J \text{ and } j+1,k} = p_{I \text{ and } i+1,j,j+1} - p_{I \text{ and } i+1,j,k} + p_{I \text{ and } i+1,j+1,k} \tag{63}$$

$$\begin{aligned} p_{I,J \text{ and } j+1,k,k+1} &= p_{I \text{ and } i+1,j,j+1,k+1} - p_{I \text{ and } i+1,j,j+1,k} \\ &\quad + p_{I \text{ and } i+1,j+1,k,k+1} - p_{I \text{ and } i+1,j,k,k+1} \\ &\quad + p_{I \text{ and } i+1,j,j+1,k,k+1} \end{aligned} \tag{64}$$

A rapid inspection shows that application of $\mathcal{L}$ term by term in 57 gives the correct result. Moreover, one sees that $\mathcal{L}[C_N^{(1)}] = C^{(2)}$ and $\mathcal{L}[C^{(2)}] =$

$C^{(3)} + C_N^{(2)}$ and similar relations hold at any order:

$$\mathcal{L}\left[C_N^{(n)}\right] = C^{(n+1)} \tag{65}$$

$$\mathcal{L}\left[C^{(n)}\right] = C^{(n+1)} + C_N^{(n)} \tag{66}$$

A couple of polynomials (P_n, Q_n) can be associated to S_n: P_n represents the C terms and Q_n the C_N terms, with the mapping $C^{(m)} \leftrightarrow X^m$. Thus,

$$(P_1, Q_1) = (X, 0) \tag{67}$$

$$(P_2, Q_2) = (X^2, X) \tag{68}$$

$$(P_3, Q_3) = (X^3 + X^2, X^2) \tag{69}$$

$$\dots \tag{70}$$

$$(P_{n+1}, Q_{n+1}) = (XP_n + XQ_n, P_n) \tag{71}$$

whence $P_{n+1} = XP_n + XP_{n-1}$. Symbolically, this recursion can be solved for P_n, owing to $P_1 = X$, $P_2 = X^2$:

$$P_n = \frac{X}{\sqrt{X^2+4X}}\left[\left(\frac{X+\sqrt{X^2+4X}}{2}\right)^n - \left(\frac{X-\sqrt{X^2+4X}}{2}\right)^n\right] \tag{72}$$

Q_n is simply deduced from the formula $Q_n = P_{n-1}$. Symbolically, it yields

$$\sum_{n=1}^{\infty} P_n \tau^n = \frac{X\tau}{1 - X(\tau + \tau^2)} \tag{73}$$

$$= \frac{1}{1+\tau}\sum_{n=1}^{\infty}(\tau+\tau^2)^n X^n \tag{74}$$

and

$$\sum_{n=1}^{\infty} Q_n \tau^n = \frac{\tau}{1+\tau}\sum_{n=1}^{\infty}(\tau+\tau^2)^n X^n \tag{75}$$

From these results is derived a rewriting of $G(\mu)$:

$$G(\mu) = 1 + \frac{1}{1+\tau}\sum_{n=1}^{\infty}(\tau+\tau^2)^n C^{(n)} + \frac{\tau}{1+\tau}\sum_{n=1}^{\infty}(\tau+\tau^2)^n C_N^{(n)} \tag{76}$$

$$= 1 + \frac{1}{\tau+1}\left[(\tau+\tau^2)\sum_i c_{i,i+1} + (\tau+\tau^2)^2\sum_{i<j} c_{i,i+1,j,j+1} + \cdots\right] + \frac{\tau}{\tau+1}\left[(\tau+\tau^2)\sum_i c_{i,i+1,N} + (\tau+\tau^2)^2\sum_{i<j} c_{i,i+1,j,j+1,N} + \cdots\right] \tag{77}$$

The second sum can be slightly transformed : it can be shown (see Appendix A) that

$$c_{i_1,i_1+1,i_2,i_2+1,\ldots,i_n,i_n+1,N} = c_{i_1,i_1+1,i_2,i_2+1,\ldots,i_n,i_n+1} - c_{i_1,i_1+1,i_2,i_2+1,\ldots,i_n,N} + \cdots - c_{i_1,i_2,i_2+1,\ldots,i_n+1,N} + c_{i_1+1,i_2,i_2+1,\ldots,i_n,i_n+1,N} \tag{78}$$

whence

$$C_N^{(n)} = \sum_{0\leqslant i_1<i_2<\ldots<i_n} c_{i_1,i_1+1,i_2,i_2+1,\ldots,i_n,i_n+1} - \sum_{0<i_2<\ldots<i_n} c_{0,i_2,i_2+1,\ldots,i_n,i_n+1,N}$$

$$= C^{(n)} - C_N^{(n-1)} + \sum_{i_3<\ldots<i_n} c_{1,i_3,\ldots,i_n,i_n+1,N} \tag{79}$$

(otherwise explicitely stated, let us recall that the sums start at site 0). Whence

$$\sum_{n=1}^{\infty}(\tau+\tau^2)^n C_N^{(n)} = -\frac{\tau+\tau^2}{1+\tau+\tau^2} + \frac{1}{1+\tau+\tau^2}\sum_{n\geqslant 1}(\tau+\tau^2)^n C^{(n)} + \frac{(\tau+\tau^2)^2}{1+\tau+\tau^2}\sum_{n\geqslant 0}(\tau+\tau^2)^n C_{1(n)N} \tag{80}$$

where

$$C_{1(n)N} = \sum_{0<i_1<i_2<\ldots<i_n} c_{1,i_1,i_1+1,i_2,i_2+1,\ldots,i_n,i_n+1,N} \tag{81}$$

As a result, one has

$$G(\mu)=\frac{1+\tau}{1+\tau+\tau^2}\left[1+\sum_{n\geqslant 1}(\tau+\tau^2)^n C^{(n)}+\tau^3\sum_{n\geqslant 0}(\tau+\tau^2)^n C_{1(n)N}\right] \tag{82}$$

This rewriting is fruitful, since the sums are directly related to the Pfaffian theory: as explained in the Appendix A, we can write

$$G(\mu)=\frac{1+\tau}{1+\tau+\tau^2}\sqrt{\det[1+A(\eta,\eta')C]} \text{ with} \tag{83}$$

$$\eta=-\tau-\tau^2 \tag{84}$$

$$\eta'=\frac{\tau^2}{1+\tau} \tag{85}$$

(the definitions of A and C are given in the Appendix A).

4.3. Asymptotic Behaviour of *G*

In ref. 8, the authors showed that the leading term of $\log\det[1+A(\eta,\eta')C]$ remains unchanged if we put $\eta'=0$. In fact, in their case, a difficulty arises, due to the fact that the subdominant term (where η' appears) diverges in a certain range of parameters and provides an analytic continuation of the persistence exponent. Here, the computation of the subdominant term cannot be performed exactly because the matrix AC does not have an exact continuous limit, but we think that it is precisely this fact that prevents the subdominant term to interfere in the analytic properties of the ldf. Anyway, we did not find any analytical problem in the ldf (see below), and interpreted this as a "gentle" behaviour of the subdominant term. In the following the replacement $\eta'=0$ is implicitely made.

Using $\log\det(M)=\mathrm{Tr}\log(M)$, we get

$$\log G(\mu)\sim-\frac{1}{2}\sum_{p=1}^{\infty}\frac{(-1)^p}{p}\mathrm{Tr}\left[(A(\eta,0)C)^p\right] \tag{86}$$

From the expressions of A and C, we have:

$$\mathrm{Tr}[(A(\eta,0)C)^p]\underset{N\to\infty}{\sim}\left(\frac{8}{\pi}(\tau+\tau^2)\right)^p\sum_{1\leqslant i_1,\ldots,i_p\leqslant N}\frac{i_1 i_2\cdots i_p}{(i_1^2+i_2^2)(i_2^2+i_3^2)\cdots(i_p^2+i_1^2)} \tag{87}$$

We get this first expression by assuming that the leading term is obtained when all indices are large (cf. ref. 8 and see below). We assume temporarily that this term is $O(\log N)$; thus we have necessarily for any $a > 1$

$$\sum_{i_1,\dots,i_p \leqslant N} \frac{i_1 i_2 \cdots i_p}{(i_1^2+i_2^2)(i_2^2+i_3^2)\cdots(i_p^2+i_1^2)}$$

$$\sim \frac{\log N}{\log a} \sum_{N \leqslant i_1,\dots,i_p \leqslant Na} \frac{i_1 i_2 \cdots i_p}{(i_1^2+i_2^2)(i_2^2+i_3^2)\cdots(i_p^2+i_1^2)} \tag{88}$$

$$\sim \frac{\log N}{\log a} \int_0^{\log a} \frac{du_1 \cdots du_p}{(1+e^{2(u_2-u_1)})(1+e^{2(u_3-u_2)})\cdots(1+e^{2(u_1-u_p)})} \tag{89}$$

Let us now consider the large a limit in the integral and perform the change of variable $v_1 = u_1$, $v_2 = u_2 - u_1, \dots v_p = u_p - u_{p-1}$ (its Jacobian is 1). Defining $s(x) = e^{-x}/(1+e^{-2x}) = 1/2\cosh(x)$, we have

$$\int_0^{\log a} \frac{du_1 \cdots du_p}{(1+e^{2(u_2-u_1)})\cdots(1+e^{2(u_1-u_p)})}$$

$$= \int_{-\infty}^{\infty} dy \int_0^{\log a} dv_1 \int dv_2 \cdots dv_p \delta(v_2 + \cdots + v_p - y) s(v_2)\cdots s(v_p) s(y) \tag{90}$$

$$\underset{a\to\infty}{\sim} \log a \times \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[\int_{-\infty}^{\infty} dv e^{ikv} s(v)\right]^p \tag{91}$$

Before proceeding further, let us show that this term can be expressed differently, which will make the $\log N$ dependence explicit:

$$\sum_{i_1,\dots i_p \leqslant N} \frac{i_1 i_2 \dots i_p}{(i_1^2+i_2^2)(i_2^2+i_3^2)\dots(i_p^2+i_1^2)}$$

$$\sim p \sum_{i_p=1}^{N} \frac{1}{i_p^p} \sum_{i_1,\dots,i_{p-1} \leqslant i_p} \frac{(i_1/i_p)\cdots(i_{p-1}/i_p)}{((i_1/i_p)^2+(i_2/i_p)^2)\cdots((i_1/i_p)^2+1)} \tag{92}$$

$$\sim p \log(N) \int_0^1 dx_1 \cdots dx_{p-1} \frac{x_1 \cdots x_{p-1}}{(x_1^2+x_2^2)\cdots(x_{p-1}^2+1)(1+x_1^2)} \tag{93}$$

(because the general term of the i_p sum is asymptotically equivalent to the integral times $1/i_p$). This expression is however not as useful as the preceding, for it leads to an expression involving a solution of a Wiener–Hopf problem.

Coming back to $G(\mu)$, Eq. (91) and ref. 8 yield

$$\log G(\mu) \sim \log(N) \times \int_{-\infty}^{\infty} \frac{dk}{4\pi} \log\left[1 + \frac{8}{\pi}(\tau + \tau^2) \int_{-\infty}^{\infty} dv\, e^{ikv} s(v)\right] \quad (94)$$

$$\Rightarrow g(\mu) = \frac{1}{8} - \left[\frac{\sqrt{2}}{\pi} \text{Arccos}\left(\frac{e^{-2\mu}}{\sqrt{2}}\right)\right]^2 \quad (95)$$

We verify that $g(0) = 0$, $-g'(0) = \langle E/\log N\rangle = 2/\pi$ and $g''(0) = \frac{8}{\pi^2}(\pi - 2) = (\langle E^2\rangle - \langle E\rangle^2)/\log N$, as it must be. The third derivative of g is $64(3-\pi)/\pi^2$ on zero. In the Fig. 4, the aspect of $f(\zeta)$ is shown and its shape is compared with a parabola. The fluctuations above the mean value are comparatively more probable than those below the mean value: intuitively a positive fluctuation of E regresses by annihilation pair by pair of the (excess) domain walls, and can be sustained as soon as the domain walls avoid each other in their erratic motion; on the contrary, a important negative fluctuation (i.e. $E < \langle E\rangle$) will recede by energy refilling from the boundary, a process which is unavoidable if the population of domain

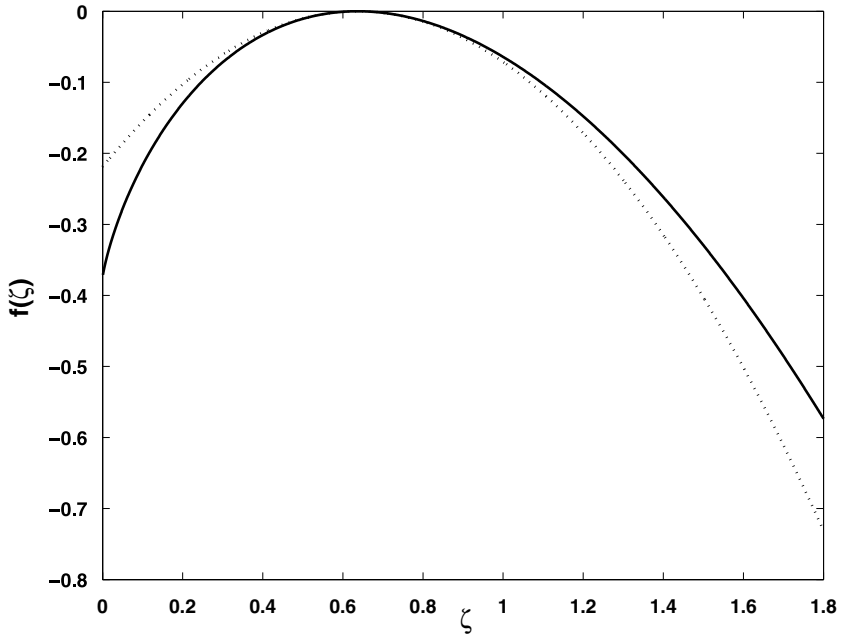

Fig. 4. The large deviation function $f(\zeta)$ for the Ising model (solid line) and the best parabolic approximation $-(\zeta - 2/\pi)^2/(2g''(0))$ (dots). The ldf does not continue in the range $\zeta < 0$.

walls is substantially lowered. This could explain qualitatively the counter-clockwise tilt of the curve.

As regards the right asymptotics of $f(\zeta)$, it is easily established (with the formula 94 which is the analytic continuation of $g(\mu)$) that $f(\zeta) \sim -(\pi\zeta/8)^2$; this explains the vaguely parabolic look of f.

Finally let us notice that the $f(\zeta)$ stops abruptly at $\zeta = 0$; this means that the probability of observing an arbitrary small energy E decreases like $N^{-3/8}$ (up to a $1/\sqrt{\log N}$ term); it illustrates the fact that an important "clearance" of the system due to an efficient (in dissipation) fluctuation is not particularly unlikely. In ref. 10 a same kind of ldf with a nondiverging left branch was found in a very different context.

4.4. Generalization to (an)isotropic Potts Models

The above calculations can be extended without difficulty to a q-states Potts model (driven by a "voter-model" dynamics: in $[t, t+dt]$, each spin imitates its left/right neighbour with a probability dt and does nothing with probability $1-2\,dt$) with an arbitrary input at σ_0 (the only requirement concerning this input is that it must be statistically stationary and have a finite time scale; we call this arbitrariness "anisotropic" for the different colours of the Potts model are not necessarily equiprobable). The energy is now defined as $e_j = 2(1-\delta_{\sigma_j,\sigma_{j+1}})$ (we put the factor two to make the Ising model equivalent to the $q=2$ Potts). If $P_0(\sigma_0)$ is the stationary probability of the spin σ_0, let us define

$$b = 2 \times \left(1 - \sum_{\ell=1}^{q} P_0(\ell)^2\right) \tag{96}$$

($b=1$ in the Ising model; $b=2$ corresponds to a $A+A \to A$ model).

Actually, this generalization does not modify deeply the preceding analysis: the reasoning on the coalescing walkers is unaffected by the multiplication of the "colours", since either the walkers collapse, which implies automatically that they "carried" the same colour (remind that the walkers cannot be considered as domain walls or vice-versa), or they do not collapse, and in that case their colours are statistically determined by the σ_0 process, which can be considered as before as an ultrafast process.

It is easily shown that $\langle E \rangle \sim \log(N) 2b/\pi$ and similarly the function G is obtained from the preceding by making the replacement $\tau \to b\tau$:

$$g(\mu) = \frac{1}{8} - \left[\frac{\sqrt{2}}{\pi} \mathrm{Arccos}\left(\frac{be^{-2\mu} + 1 - b}{\sqrt{2}}\right)\right]^2 \tag{97}$$

In the Fig. 5 were plotted the ldf for several values of b. To compare these functions we normalized the coordinates, so that their maximum is located at $2/\pi$ and their curvature at the maximum is -1. Once normalized, these functions look very similar, and the physical informations are mainly contained in the first two moments: we have $\langle E\rangle/\log N \sim 2b/\pi$ and $(\langle E^2\rangle - \langle E\rangle^2)/\log N \sim 4b[\pi + (\pi - 4)b]/\pi^2$. As a result, irrespective of the details of the injection, the anisotropic Potts models verify asymptotically the simple relation

$$\frac{\Delta E^2}{\langle E\rangle} = \left[(\pi - 4)\frac{\langle E\rangle}{\log N} + 2\right] \tag{98}$$

A natural question would be to know if such a relation can still be verified if intrinsic dynamics, although dissipative, is not equivalent to a voter model.

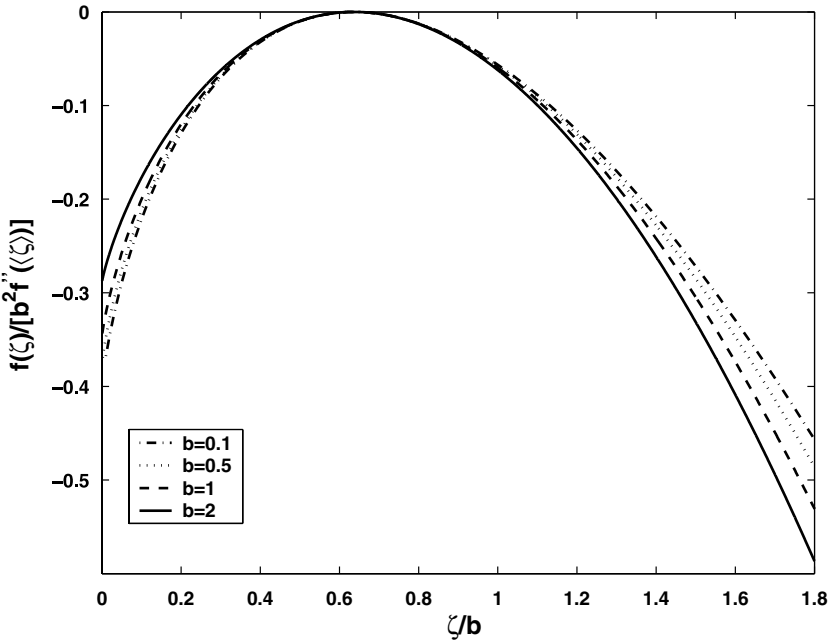

Fig. 5. Normalized ldf for different values of b for the anisotropic Potts models: The plots are $f(\zeta)/[b^2 f''(\langle\zeta\rangle)]$ as a function of ζ/b. The Ising model corresponds to $b = 1$.

5. NONEQUILIBRIUM "FREE ENERGY"

As it was possible to compute the ldf of E, we can ask whether a more precise description of the fluctuations of the energy field could be tractable. The first idea is to form n_0 groups of N consecutive spins and try to compute the probability to observe a certain coarse-grained energy profile $(E_1, \ldots, E_{n_0})$ when N is large. But this idea does not take into account the great intrinsic inhomogeneity of the dissipative systems: if the typical energy of the N first spins is $O(\log N)$, the typical energy of the spins from indices N to $2N$ is only $O(1)$, which prevents to give a satisfactory description in terms of ldf. The second idea is to rectify by hand this inhomogeneity, considering for instance variables like $\sum_i i e_i$. But it is easy to show that they do not obey a large deviation theorem in the large N limit. Actually the only way to construct a ldf capable of describing more precisely the energy fluctuations is to consider *unequal* blocks of spins: the first group gather spins σ_0 to σ_{N-1}, the second spins from σ_N to σ_{N^2-1}, ..., the last, spins from $\sigma_{N^{n_0-1}}$ to $\sigma_{N^{n_0}-1}$. We are going to show that this inhomogeneous grouping of spins allows a relevant description in terms of ldf.

For the sake of simplicity, we come back to the Ising model, but the generalization of the preceding paragraph could be processed along similar simple lines. As well, we forget for a moment the particular way to group the spins together and consider something more general:

$$H(\boldsymbol{\mu}) = \left\langle \exp\left(- \sum_{i=0}^{M-1} \mu_i (1 - \sigma_i \sigma_{i+1}) \right) \right\rangle \tag{99}$$

H is a characteristic function for M spins where μ is replaced by an arbitrary "field" $\boldsymbol{\mu}$. A multidimensional Laplace inversion performed over $\boldsymbol{\mu}$ would give the probability of the energy density field.

A similar procedure as for G leads to

$$H(\boldsymbol{\mu}) = 1 + \sum_i \tau_i p_I + \sum_{i<j} \tau_i \tau_j p_{I,J} + \cdots \tag{100}$$

$$\equiv 1 + T_1 + T_2 + \cdots \tag{101}$$

with $\tau_j = (e^{-2\mu_j} - 1)/2$. The resummation procedure is more complicated than for G. Let us define $\Delta_j = \tau_{j-1} - \tau_j$. The expansion of T_2 using the recursion relation (A.1) leads to

$$T_2 = \sum_{i<j} \tau_i \Delta_j p_{I \text{ and } i+1,j} + \sum_{i<j} \tau_i \tau_j p_{I \text{ and } i+1,j,j+1} \tag{102}$$

Symbolically, the "τ" term of T_1 is transformed to a "$\tau\Delta + \tau\tau$" term at the next step. A lengthy computation shows that

$$\begin{aligned} T_3 = & \sum_{i<j<k} \tau_i \Delta_j \Delta_k p_{I \text{ and } i+1,j,k} + \sum_{i<j<k} \tau_i \Delta_j \tau_k p_{I \text{ and } i+1,j,k,k+1} \\ & + \sum_{i<k} \tau_i \tau_k^2 p_{I \text{ and } i+1,k,k+1} + \sum_{i<j<k} \tau_i \tau_j \Delta_k p_{I \text{ and } i+1,j,j+1,k} \\ & + \sum_{i<j<k} \tau_i \tau_j \tau_k p_{I \text{ and } i+1,j,j+1,k,k+1} \qquad (103) \\ = & \,''\tilde{\tau}\tilde{\Delta}\tilde{\Delta} + \tilde{\tau}\tilde{\Delta}\tilde{\tau} + \tilde{\tau}\tilde{\tau}_2 + \tilde{\tau}\tilde{\tau}\tilde{\Delta} + \tilde{\tau}\tilde{\tau}\tilde{\tau}'' \qquad (104) \end{aligned}$$

where the last line is a symbolic rewriting of T_3. Besides, the computation shows that $(\ldots)\tilde{\Delta}$ is transformed into $(\ldots)(\tilde{\Delta}\tilde{\Delta} + \tilde{\Delta}\tilde{\tau} + \tilde{\tau}_2)$. Thus, as for G, these simple rules allow us to generate any order, which are rapidly increasing in complexity:

$$\begin{aligned} T_3 = & \tilde{\tau}\tilde{\Delta}\tilde{\tau} + \tilde{\tau}\tilde{\Delta}\tilde{\Delta} + \tilde{\tau}\tilde{\tau}_2 + \tilde{\tau}\tilde{\tau}\tilde{\Delta} + \tilde{\tau}\tilde{\tau}\tilde{\tau} \qquad (105) \\ \Rightarrow T_4 = & \tilde{\tau}\tilde{\Delta}\tilde{\tau}\tilde{\Delta} + \tilde{\tau}\tilde{\Delta}\tilde{\tau}\tilde{\tau} + \tilde{\tau}\tilde{\Delta}\tilde{\Delta}\tilde{\Delta} + \tilde{\tau}\tilde{\Delta}\tilde{\Delta}\tilde{\tau} + \tilde{\tau}\tilde{\Delta}\tilde{\tau}_2 \\ & + \tilde{\tau}\tilde{\tau}_2\tilde{\Delta} + \tilde{\tau}\tilde{\tau}_2\tilde{\tau} + \tilde{\tau}\tilde{\tau}\tilde{\Delta}\tilde{\tau} + \tilde{\tau}\tilde{\tau}\tilde{\Delta}\tilde{\Delta} + \tilde{\tau}\tilde{\tau}\tilde{\tau}_2 + \tilde{\tau}\tilde{\tau}\tilde{\tau}\tilde{\Delta} + \tilde{\tau}\tilde{\tau}\tilde{\tau}\tilde{\tau} \qquad (106) \end{aligned}$$

These terms can be grouped into four categories: there are terms (called "u_n" in the following) without any $\tilde{\Delta}$, terms with a single and terminal $\tilde{\Delta}$ (called "v_n"), terms like $\tilde{\tau}\tilde{\Delta}\tilde{\Delta}\tilde{\tau}$ with multiple internals $\tilde{\Delta}$ but not ending with a $\tilde{\Delta}$ (called "w_n"), and terms with multiple $\tilde{\Delta}$ ending with a $\tilde{\Delta}$ (called "z_n"). Recursions relations exist between these fours sequences: one has for u and v:

$$u_1 = \tilde{\tau}, \quad u_2 = \tilde{\tau}\tilde{\tau}, \quad u_{n+1} = u_n\tilde{\tau} + u_{n-1}\tilde{\tau}_2 \qquad (107)$$

$$v_1 = 0, \quad v_{n+1} = u_n\tilde{\Delta} \qquad (108)$$

Let us recall that

- the elements $\tilde{\tau}$, $\tilde{\tau}_2$ and $\tilde{\Delta}$ of the symbolic algebra do not commute.
- the element $\tilde{\tau}_2$ is not equivalent to $\tilde{\tau}\tilde{\tau}$

Formally, one can sum up the u_n and v_n terms:

$$U \equiv \sum_{n=1}^{\infty} u_n = \tilde{\tau}(1-\tilde{\tau}-\tilde{\tau}_2)^{-1} \tag{109}$$

$$V \equiv \sum_{n=1}^{\infty} v_n = \tilde{\tau}(1-\tilde{\tau}-\tilde{\tau}_2)^{-1}\widetilde{\Delta} \tag{110}$$

The w and z terms obey also recursion relations:

$$w_{n+1} = v_n\widetilde{\tau} + z_n\tilde{\tau} + w_n\tilde{\tau} + z_n\widetilde{\not{\Delta}}\tilde{\tau}_2 \tag{111}$$

$$z_{n+1} = (w_n + z_n + v_n)\widetilde{\Delta} \tag{112}$$

where $z_n\widetilde{\not{\Delta}}$ means "z_n without its final $\widetilde{\Delta}$". Defining $W = \sum_{n\geqslant 3} w_n$ and $Z = \sum_{n\geqslant 3} z_n$, a simple computation gives

$$H(\boldsymbol{\mu}) = 1 + U + V + W + Z \tag{113}$$

$$= 1 + \tilde{\tau}\left(1-\widetilde{\tau}-\widetilde{\tau}_2-\widetilde{\Delta}\right)^{-1} \tag{114}$$

This simple formal result hides a real complexity: let us recall that for instance

$$\tilde{\tau}\widetilde{\Delta}\tilde{\tau}_2\widetilde{\Delta} = \sum_{i<j<k<\ell} \tau_i \Delta_j \tau_k^2 \Delta_\ell c_{i,i+1,j,k,k+1,\ell} \tag{115}$$

A useful rewriting of H is

$$H(\boldsymbol{\mu}) = 1 + \tilde{\tau}\sum_{p=0}^{\infty}\left[(1-\tilde{\tau}-\tilde{\tau}_2)^{-1}\widetilde{\Delta}\right]^p (1-\tilde{\tau}-\tilde{\tau}_2)^{-1} \tag{116}$$

5.1. Recovering *G*

If all the μ_j are constant, this sum is considerably simplified: in fact, all Δ_j but $\Delta_N = \tau$ are zero, and $\tilde{\tau}_2 = \tau\tilde{\tau}$ such that

$$H(\mu) = 1 + \tilde{\tau}(1-\tilde{\tau}-\tilde{\tau}_2)^{-1} + \tilde{\tau}(1-\tilde{\tau}-\tilde{\tau}_2)^{-1}\widetilde{\Delta} \tag{117}$$

$$= 1 + \tau\sum_{p=0}^{\infty}(\tau+\tau^2)^p C^{(p+1)} + \tau^2\sum_{p=0}^{\infty}(\tau+\tau^2)^p C_N^{(p+1)} = G(\mu) \tag{118}$$

$G(\mu)$ is recovered.

5.2. Two Groups of Spins

Let us now consider a situation with two clusters of μ_j: $\mu_j = \mu$ if $j \in [0, N-1]$ and $\mu_j = \mu'$ if $j \in [N, N^2-1]$. In that case, two Δ_j only are non-zero ($\Delta_N = \tau - \tau'$ and $\Delta_{N^2} = \tau'$) and H is expressed as

$$H(\mu, \mu') = 1 + \tilde{\tau}(1 - \tilde{\tau} - \tilde{\tau}_2)^{-1} + \tilde{\tau}(1 - \tilde{\tau} - \tilde{\tau}_2)^{-1}\tilde{\Delta}(1 - \tilde{\tau} - \tilde{\tau}_2)^{-1} + \tilde{\tau}(1 - \tilde{\tau} - \tilde{\tau}_2)^{-1}\tilde{\Delta}(1 - \tilde{\tau} - \tilde{\tau}_2)^{-1}\tilde{\Delta} \tag{119}$$

We can write $H(\mu, \mu')$ as $H(\mu, \mu') = H(0, \mu') + \Delta H$. It is easy to see that $H(0, \mu')$ (resp. ΔH) is made with all terms where the first τ_i of the sums is τ' (resp. τ). Defining

$$\tilde{H}(\mu, \mu') \equiv 1 + (\tilde{\tau} + \tilde{\tau}_2)(1 - \tilde{\tau} - \tilde{\tau}_2)^{-1} + (\tilde{\tau} + \tilde{\tau}_2)(1 - \tilde{\tau} - \tilde{\tau}_2)^{-1}\tilde{\Delta}(1 - \tilde{\tau} - \tilde{\tau}_2)^{-1} + (\tilde{\tau} + \tilde{\tau}_2)(1 - \tilde{\tau} - \tilde{\tau}_2)^{-1}\tilde{\Delta}(1 - \tilde{\tau} - \tilde{\tau}_2)^{-1}\tilde{\Delta} \tag{120}$$

we get

$$\tilde{H} = 1 + (1 + \tau)\Delta H + (1 + \tau')[H(0, \mu') - 1] \tag{121}$$

$$= -\tau' + (1 + \tau)H + (\tau' - \tau)H(0, \mu') \tag{122}$$

$\tilde{H}$ is interesting for it is easier to compute than H; the last expression shows that once $\tilde{H}$ is computed, so is H, since $H(0, \mu')$ can be estimated along lines similar to those we followed for G.

Let us define $\xi = \tau + \tau^2$ and $\xi' = \tau' + \tau'^2$. Similarly $\tilde{\xi}$ (resp. $\tilde{\xi}'$) is the operator $\tilde{\tau} + \tilde{\tau}_2$ applied on indices less (resp. greater) than N. As an example,

$$\tilde{\xi}^2\tilde{\xi}' = \xi^2\xi' \sum_{0 \leqslant i_1 < i_2 < N \leqslant j_1} c_{i_1 i_1+1, i_2, i_2+1, j_1, j_1+1} \tag{123}$$

Similarly $\tilde{\Delta}_N$ and $\tilde{\Delta}_{N^2}$ term the $\tilde{\Delta}$ operator where the index of application is precised. We have

$$\tilde{H}(\mu, \mu') = 1 + \sum_{p+q>0} \tilde{\xi}^p\tilde{\xi}'^q + \sum_{p \geqslant 1, q \geqslant 0} \tilde{\xi}^p\tilde{\Delta}_N\tilde{\xi}'^q + \sum_{p+q>0} \tilde{\xi}^p\tilde{\xi}'^q\tilde{\Delta}_{N^2} + \sum_{p \geqslant 1, q \geqslant 0} \tilde{\xi}^p\tilde{\Delta}_N\tilde{\xi}'^q\tilde{\Delta}_{N^2} \tag{124}$$

Terms with a single $\tilde{\Delta}$ can be further simplified using (79):

$$\tilde{\xi}^p \widetilde{\Delta}_N \tilde{\xi}'^q = \xi^p \xi'^q \Delta_N \left[-C_{(p),N,(q-1),N^2} + C_{(p),(q)} - C_{(p-1),N,(q)} + C_{1,(p-2),N,(q)} \right]$$

$$\tilde{\xi}^p \tilde{\xi}'^q \widetilde{\Delta}_{N^2} = \xi^p \xi'^q \Delta_{N^2} \left[C_{(p),(q)} - C_{(p),N,(q-1),N^2} + C_{(p-1),N,(q),N^2} \right. \tag{125}$$

$$\left. - C_{(p-1),(q),N^2} + C_{1,(p-2),(q),N^2} \right] \tag{126}$$

The definition of the symbols $C_{\dots}$ is generalized from (81); $C_{(p),(q)}$ deserves however a precision: it holds for a summation over $0 \leqslant i_1 < \cdots < i_p < N \leqslant j_1 < \cdots < j_q < N^2$ (instead of $\cdots < N < j_1 < \cdots$). After a computation, we get

$$\begin{aligned} \widetilde{H}(\mu, \mu') = & \frac{(1+\tau)^2}{1+\xi} \sum_{p+q>0} \left(\tilde{\xi}^p \tilde{\xi}'^q + \tilde{\xi}^p \widetilde{\Delta}_N \tilde{\xi}'^q \widetilde{\Delta}_{N^2} \right) \\ & + \frac{\xi^2}{1+\xi} \sum_{p+q \geqslant 0} \xi^p \xi'^q [\Delta_N C_{1,(p),N,(q)} + \Delta_{N^2} C_{1,(p),(q),N^2}] \\ & - \Delta_N \sum_{q \geqslant 1} \tilde{\xi}'^q + \tau' \sum_{q \geqslant 1} \widetilde{\Delta}_N \tilde{\xi}'^q \widetilde{\Delta}_{N^2} + \text{boundary terms} \end{aligned} \tag{127}$$

whence

$$H(0, \mu') = \widetilde{H}(0, \mu')/(1+\tau') = \sum_{q>0} \left(\tilde{\xi}'^q - \tilde{\tau}' \tilde{\xi}'^q \widetilde{\Delta}_{N^2} \right) + \text{b.t.} \tag{128}$$

and finally

$$\begin{aligned} H(\mu, \mu') = & \frac{1+\tau}{1+\xi} \sum_{p+q>0} \left(\tilde{\xi}^p \tilde{\xi}'^q + \tilde{\xi}^p \widetilde{\Delta}_N \tilde{\xi}'^q \widetilde{\Delta}_{N^2} \right) \\ & + \frac{\tau \xi}{1+\xi} \sum_{p+q \geqslant 0} \xi^p \xi'^q [\Delta_N C_{1,(p),N,(q)} + \Delta_{N^2} C_{1,(p),(q),N^2}] + \text{b.t.} \end{aligned} \tag{129}$$

When $\tau = \tau'$, the formula for G is again recovered. It is interesting to note that H has here again a Pfaffian structure: if B is the skew-symmetric $N^2 \times N^2$ matrix where all elements but $b_{i,i+1} = -\tau_i(\tau_i + 1)$, $b_{1,N} = -\tau^2 \Delta_N$, $b_{1,N^2} = -\tau^2 \Delta_{N^2}$, $b_{N,N^2} = -\Delta_N \Delta_{N^2}$ are zero, then

$$H(\mu, \mu') = \frac{1+\tau}{1+\tau+\tau^2} \sqrt{\det(1+BC)} + \text{b.t.} \tag{130}$$

By induction from the preceding case, we assume that the ldf can be computed as if $\Delta_N = \Delta_{N^2} = 0$. In that case, we get

$$\log H \simeq -\frac{1}{2}\sum_{p=1}^{\infty}\left(-\frac{8}{\pi}\right)^p \frac{1}{p} B_p \tag{131}$$

where B_p is a trace term (we omit the p summation symbols):

$$B_1 \simeq \xi_{i_1}\frac{1}{2i_1} \tag{132}$$

$$B_p \simeq \xi_{i_1}\cdots\xi_{i_p}\frac{i_1 i_2\cdots i_p}{(i_1^2+i_2^2)(i_2^2+i_3^2)\cdots(i_p^2+i_1^2)} \tag{133}$$

5.3. General Case

For the general case, where n_0 groups of spins are considered instead of two, the generalization is quite immediate and similar conclusions hold, that is, H is proportional to a Pfaffian (up to boundary terms) whose leading term is unchanged if off-second-diagonal terms are replaced by zero.

As for the computation of G, there are two different ways to compute the leading term of B_p. For the inhomogeneous case, it is convenient to write

$$\begin{aligned} B_p \underset{N\to\infty}{\simeq}\ & p\sum_{i_p=0}^{N^{n_0}-1}\xi_{i_p}\sum_{i_1,i_2,\ldots,i_{p-1}=1}^{i_p}\xi_{i_1}\cdots\xi_{i_{p-1}} \\ & \times\frac{(i_1/i_p)(i_2/i_p)\cdots(i_{p-1}/i_p)i_p^p}{([i_1/i_p]^2+[i_2/i_p]^2)([i_2/i_p]^2+[i_3/i_p]^2)\cdots(1+[i_1/i_p]^2)i_p^{2p}} \qquad (134) \\ \simeq\ & p\sum_{i_p=0}^{N^{n_0}-1}\frac{\xi_{i_p}}{i_p}\int_0^1 dx_1 \\ & \times\int_0^1 dx_2\cdots\int_0^1 dx_{p-1}\ \xi_{i_p x_1}\cdots\xi_{i_p x_{p-1}}\frac{x_1 x_2\cdots x_{p-1}}{(x_1^2+x_2^2)\cdots(x_{p-1}^2+1)(1+x_1^2)} \qquad (135) \end{aligned}$$

Let us recall that we consider a particular profile where μ_i (i.e. ξ_i) is constant in each interval $[0, N-1]$, $[N, N^2-1], \ldots, [N^{n_0-1}, N^{n_0}-1]$, that is $\xi_i = \xi(E[\log(i)/\log(N)])$. Thus, $\xi_{i_p x_1} = \xi(E[\log(i_p)/\log(N)+$

$\log(x_1)/\log(N)]$) and $\xi_{i_p x_1} = \xi_{i_p}$ except for x_1 of order $1/N$. As a result, in this case,

$$B_p \simeq p \sum_{i_p=0}^{N^n-1} \frac{\xi_{i_p}^p}{i_p} \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_{p-1} \frac{x_1 x_2 \cdots x_{p-1}}{(x_1^2+x_2^2)\cdots(x_{p-1}^2+1)(1+x_1^2)} \tag{136}$$

$$\sim p(\log N)(\xi(0)^p + \xi(1)^p + \cdots + \xi(n-1)^p) \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_{p-1} \times \frac{x_1 x_2 \cdots x_{p-1}}{(x_1^2+x_2^2)\cdots(x_{p-1}^2+1)(1+x_1^2)} \tag{137}$$

Thus, the coupling vanishes at the level of the ldf, and we have (the vector $\boldsymbol{\mu} = (\mu(1), \ldots, \mu(n_0))$ holds here for the n_0 different values of μ for the n_0 groups of spins)

$$g(\boldsymbol{\mu}) = \sum_{i=1}^{n_0} \left(\frac{1}{8} - \frac{2}{\pi^2} \left[\operatorname{Arccos}\left(\frac{e^{-2\mu(i)}}{\sqrt{2}} \right) \right]^2 \right) \tag{138}$$

Thus, it appears that the successive groups of spins are *decorrelated* at the level of the ldf. This decoupling shows that the situation is here completely different from those arising in conservative systems, and sheds light on the intimate relation between the *conservative* character of an observable and the appearance of severe correlations in a NESS. Here any fluctuation arising somewhere in the system affects only marginally its vicinity, for this fluctuation is mainly locally destroyed by the dissipation.

6. CONCLUSION AND PERSPECTIVES

We exhibited an example of dissipative system in a stationary state where a global variable (the cumulative energy E) obeys a large deviation theorem; moreover its ldf $f(x)$ appeared to be independent of the injection mechanism. This particular "boundary layer" structure made the exact computation of f tractable, and we were also able to compute the ldf associated with the probability of a particular energy profile. Due to strong inhomogeneities in the stationary state, such a profile must be defined in a nonstandard way, and there is only one coherent way to do this. We proved also that the stationary state do not exhibit correlations as

conservative systems do when pulled out of equilibrium, for the bulk dissipation prevents the fluctuations to be transported unchanged from one place to another. Incidentally, it is interesting to mention the ref. 11, where the authors come to quite different conclusions: they study the equilibrium properties of a system driven by Kawasaki+Glauber dynamics and show that long range correlations do exist, although the system be in equilibrium; they conjecture thus that "correlations are more likely to be a generic feature of non reversible dynamics". In our case however, where internal dynamics are also non reversible, correlations are not induced. Maybe the dynamical details play an important role, and nonlocal dynamics like Kawasaki's could impose correlations at the large deviation level.

A natural extension of this work is to add a systematic drift to the system and test if it would be able to rebuild correlations; besides, if we interpret it as a convection phenomenon in a realistic system, the study of such a competition (and balance) between the dissipation and the transport in the structuration of the fluctuations could give interesting insights toward the characterization of dissipative NESS.

But some care must be taken in doing a modification of internal dynamics of a dissipative system: actually, it is quite easy to destroy the large deviation "structure" of the energy profile. In fact, the appearance of strong correlations in a stationary state seems to be slightly antithetical with a large deviation theorem, since the latter is mathematically expressed for a sum of *N independent* random variables; if we know that as a rule weak correlations do not prevent a large deviation theorem to hold, it is obvious that strongly correlated systems have no reason to verify this theorem anymore. In this respect, the results of ref. 1 seem at first sight paradoxical, since strong correlations are present in their systems; but actually, if their correlations are strong in the sense they "connect" arbitrary distant places in the system, they are nevertheless weak as regards their intensity, since they are inversely proportional to the number of particles.

APPENDIX A

The central point of this model is that probabilities $c_{\dots}$, whatever the event symbolised by the $\dots$, can be expressed in terms of the $c_{i,j}$ only. To see this, let us consider the recursion relation $c_{i,j}$ verifies:

$$c_{i,j} = \frac{1}{4}\left(c_{i+1,j} + c_{i-1,j} + c_{i,j+1} + c_{i,j-1}\right) \quad \text{if } i < j \tag{A.1}$$

$$c_{0,j} = 1 \quad \forall\ j \geqslant 0 \tag{A.2}$$

$$c_{i,i} = 0 \quad \forall\ i \geqslant 1 \tag{A.3}$$

The Eq. (A.1) comes from an explicitation of the next move of the walkers: there are 4 equiprobable moves and for instance Prob[$(i, j)|$ next move is $(i \to i+1)] = c_{i+1,j}$. The boundary conditions (A.3) ensure that this relation is valid even if $i+1=j$. Similarly the b.c. (A.2) plays the same role for the event $i \to i-1$ when $i=1$.

Very similar relations can be written for any $c_{i,j,k,\ldots}$ (the r.h.s of the recursion relation has six terms for $c_{i,j,k}$, eight terms for $c_{i,j,k,\ell}$, etc ...). It is then just a matter of calculation to verify that the solution of the recursion relation for $c_{i,j,k}$ and $c_{i,j,k,\ell}$ are

$$c_{i,j,k} = c_{i,j} + c_{j,k} - c_{i,k} \tag{A.4}$$

$$c_{i,j,k,\ell} = c_{i,j}c_{k,\ell} + c_{i,\ell}c_{j,k} - c_{i,k}c_{j,\ell} \tag{A.5}$$

These relations can be generalized to higher orders:

$$c_{i_1,i_2,\ldots,i_{2n+1}} = c_{i_1,\ldots,i_{2n}} - c_{i_1,\ldots,i_{2n-1},i_{2n+1}} + \cdots + c_{i_2,\ldots,i_{2n+1}} \tag{A.6}$$

$$c_{i_1,i_2,\ldots,i_{2n}} = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \varepsilon(\sigma) c_{i_{\sigma_1} i_{\sigma_2}} c_{i_{\sigma_3} i_{\sigma_4}} \ldots c_{i_{\sigma_{2n-1}} i_{\sigma_{2n}}} \tag{A.7}$$

In the last expression, S_{2n} is the set of permutations of $\{1, 2, \ldots, 2n\}$ and the $c_{i,j}$ are antisymmetrised: $c_{j,i} \equiv -c_{i,j}$ if $j > i$. The formula (A.7) show that the $c_{i_1,\ldots,i_{2n}}$ are Pfaffians associated with the antisymmetric matrix $C = (c_{i,j})$: as demonstrated in ref. 8, if A is the $N \times N$ matrix

$$A(\eta, \xi) = \eta \begin{pmatrix} 0 & 1 & 0 & & 0 & \xi \\ -1 & 0 & 1 & 0 & & 0 \\ 0 & -1 & 0 & 1 & 0 & \\ & & . & 0 & . & \\ 0 & & & . & & 1 \\ -\xi & 0 & & & -1 & 0 \end{pmatrix} \tag{A.8}$$

We have the relation

$$\det(\mathrm{Id} + AC) = \left[1 - \eta \sum_i c_{i,i+1} + \eta^2 \sum_{i<j} c_{i,i+1,j,j+1} - \cdots \right.$$
$$\left. - \xi \left\{ \eta c_{1,N} - \eta^2 \sum_i c_{1,i,i+1,N} + \cdots \right\} \right]^2 \tag{A.9}$$

$$= \left[1 - \eta C^{(1)} + \eta^2 C^{(2)} - \cdots - \xi \left\{ \eta C_{1(0)N} - \eta^2 C_{1(1)N} + \cdots \right\} \right]^2 \tag{A.10}$$

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